Journal of Applied Mechanics and Technical Physics, Vol. 48, No. 3, pp. 460–466, 2007

ANTIPLANE STRAIN OF A BODY UNDERGOING LARGE-ROTATIONS

V. D. Bondar'

UDC 539.3

Antiplane strain of a cylindrical elastic body undergoing large rotations under surface load in the absence of body loads is studied. The form of the elastic potential corresponding to this strain is found. The stresses, the strains, and the displacement are expressed in terms of pressure and two independent strains and the pressure is expressed in terms of the linear strain invariant. For the strains and displacement, nonlinear boundary-value problems are formulated and their ellipticity conditions are given. The linear problem for the displacement is obtained by transformation of variables. An example of determining the displacement is considered.

Key words: *displacement, Almansi strains, rotations, Cauchy stresses, elastic potential, nonlinearity, boundary-value problem.*

In a number of cases, rotations of elements of a deformed body can exceed substantially elongations and shears. This situation occurs, in particular, for deformation of flexible bodies and also massive bodies near the external and internal boundaries. For these cases, the strain–displacement relations which occupy intermediate position between the formulas of linear elasticity and the general nonlinear relations were obtained in [1]. Using these relations, we study the antiplane strain of an isotropic cylindrical body in the context of the nonlinear theory of elasticity in the actual variables x_1, x_2, and x_3 ($x_1 = x$ and $x_2 = y$ are the transverse coordinates and $x_3 = z$ is the longitudinal coordinate) assuming that body forces are absent and surface load is given.

This model is determined by equilibrium equations, Murnaghan's law, compatibility equation, relation of the strain invariants in terms of its components, and strain–displacement relations [2]. We write these relations in actual variables.

Expressing the displacement gradients $\partial_k u_l$ in terms of the symmetric component e_{kl} and asymmetric component ω_{kl}:

$$\begin{gathered}\partial_k u_l = e_{kl} + \omega_{kl} \qquad (\partial_k = \partial/\partial x_k),\\ 2e_{kl} = \partial_k u_l + \partial_l u_k, \qquad 2\omega_{kl} = \partial_k u_l - \partial_l u_k\end{gathered} \tag{1}$$

[e_{kl} are components of the linear strain tensor (elongations and shears) and ω_{kl} are the rotation-tensor components], we write Novozhilov's formulas for the Almansi strains E_{kl} as

$$2E_{kl} = 2e_{kl} - \omega_{km}\omega_{lm} \tag{2}$$

[the right side (2) contains terms of the same order of magnitude]. In Eqs. (1) and (2) and below the subscripts take the values 1, 2, and 3; summation is performed over repeated indices.

For the antiplane strain of a cylindrical body (displacement is directed along the body and does not depend on the longitudinal coordinate [3, 4]), we obtain

$$u_1 = 0, \qquad u_2 = 0, \qquad u_3 = w(x, y).$$

Novosibirsk State University, Novosibirsk 630090; bond@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 48, No. 3, pp. 191–198, May–June, 2007. Original article submitted July 4, 2006.

In accordance with (1), we have

$$e_{11} = e_{22} = e_{33} = e_{12} = 0, \qquad e_{31} = \partial_x w/2, \qquad e_{32} = \partial_y w/2,$$

$$\omega_{11} = \omega_{22} = \omega_{33} = \omega_{12} = 0, \qquad \omega_{31} = -\partial_x w/2, \qquad \omega_{32} = \partial_y w/2$$

and, hence, formulas (2) yield

$$\begin{gathered} E_{11} = -(\partial_x w)^2/8, \qquad E_{22} = -(\partial_y w)^2/8, \qquad E_{33} = ((\partial_x w)^2 + (\partial_y w)^2)/8, \\ E_{12} = -\partial_x w\,\partial_y w/8, \qquad E_{31} = \partial_x w/2, \qquad E_{32} = \partial_y w/2, \qquad E_{kl} = E_{kl}(x,y). \end{gathered} \tag{3}$$

Eliminating the displacement from (3), we obtain the finite and differential strain-compatibility conditions

$$\begin{gathered} E_{11} = -E_{31}^2/2, \qquad E_{22} = -E_{32}^2/2, \qquad E_{33} = -(E_{31}^2 + E_{32}^2)/2, \qquad E_{12} = -E_{31}E_{32}/2, \\ \frac{\partial E_{32}}{\partial x} - \frac{\partial E_{31}}{\partial y} = 0. \end{gathered} \tag{4}$$

The finite conditions allow one to express the strains in terms of the two independent components E_{31} and E_{32}, whereas the differential condition establishes a differential relation between them.

According to the equalities $2E_{31} = \partial_x w$ and $2E_{32} = \partial_y w$ in (3), the independent strains determine the displacement by quadrature

$$w = 2 \int\limits_{(x_0,y_0)}^{(x,y)} (E_{31}\,dx + E_{32}\,dy) + w_0 \qquad (w_0 = \text{const}), \tag{5}$$

in which, according to (4), the integral is path independent and the constant is the displacement specified at the boundary point.

By virtue of (4), the basic strain invariants E_k as functions of the strain-tensor components or functions of two independent strains are determined by the formulas

$$E_1 = E = E_{kk} = -(E_{31}^2 + E_{32}^2),$$

$$2E_2 = E_{kk}E_{ll} - E_{kl}E_{lk} = -2(E_{31}^2 + E_{32}^2)(1 - (E_{31}^2 + E_{32}^2)/4), \quad E_3 = \det E_{kl} = 0.$$

These relations imply the properties of the invariants

$$4E_2 = E(4+E), \quad E_3 = 0, \quad 1 - 2E_1 + 4E_2 - 8E_3 = (1+E)^2, \quad E_k = E_k(x,y), \tag{6}$$

i.e., the invariants are constant along the body and expressed in terms of the linear invariant.

For an isotropic body, the elastic potential U and the material density ρ are functions of the basic strain invariants:

$$U = U(E_1, E_2, E_3), \qquad \rho = \rho_0(1 - 2E_1 + 4E_2 - 8E_3)^{1/2}$$

(ρ_0 is the initial density). For the antiplane strain, by virtue of (6), these quantities depend only on the linear invariant:

$$U = U(E), \qquad \rho = \rho_0(1+E). \tag{7}$$

It follows that the body is compressible for the Novozhilov nonlinear model, whereas it exhibits incompressible behavior for this strain in the general geometrically nonlinear case [5].

Using Murnaghan's law

$$P_{kl} = \frac{\rho}{\rho_0}\,(\delta_{kn} - 2E_{kn})\,\frac{\partial U}{\partial E_{ln}}$$

(δ_{kn} is the Kronecker symbol), which relates the Cauchy stress P_{kl} to the Almansi strain E_{kl} and taking into account (7) and the relations

$$E = E_{ln}\delta_{nl}, \qquad \frac{\partial E}{\partial E_{ln}} = \delta_{nl}, \qquad \frac{\partial U}{\partial E_{ln}} = \frac{\partial U}{\partial E}\,\frac{\partial E}{\partial E_{ln}} = U'(E)\delta_{nl}$$

we infer that the stresses are quasilinear functions of strains dependent on the transverse coordinates:

$$P_{kl}(x,y) = -q(E)(\delta_{kl} - 2E_{kl}). \tag{8}$$

Here q is the pressure determined as

$$q(x,y) = -(1+E)U'(E). \tag{9}$$

Using (4), one expresses stresses (8) in terms of pressure and independent strains:

$$\begin{gathered} P_{11} = -q(1-2E_{11}) = -q(1+E_{31}^2), \qquad P_{12} = 2qE_{12} = -qE_{31}E_{32}, \\ P_{22} = -q(1-2E_{22}) = -q(1+E_{32}^2), \qquad P_{32} = 2qE_{32}, \\ P_{33} = -q(1-2E_{33}) = -q(1+E_{31}^2+E_{32}^2), \qquad P_{31} = 2qE_{31}. \end{gathered} \tag{10}$$

Thus, the problem of determining stresses (10), strains (4), and displacement (5) reduces to finding the pressure and two independent strains. These quantities should satisfy three equations of equilibrium and strain-compatibility equation. Below, we show that this system is compatible: the first two equations determine the pressure and the last two equations the independent strains.

Taking into account formulas (10), we write the equilibrium equations in the absence of forces ($\partial_k P_{kl} = 0$) and the strain-compatibility equation (4) as

$$(1+E_{31}^2)\frac{\partial q}{\partial x} + E_{31}E_{32}\frac{\partial q}{\partial y} + q\Big[E_{31}\Big(\frac{\partial E_{31}}{\partial x} + \frac{\partial E_{32}}{\partial y}\Big) + E_{31}\frac{\partial E_{31}}{\partial x} + E_{32}\frac{\partial E_{31}}{\partial y}\Big] = 0,$$

$$(1+E_{32}^2)\frac{\partial q}{\partial y} + E_{32}E_{31}\frac{\partial q}{\partial x} + q\Big[E_{32}\Big(\frac{\partial E_{31}}{\partial x} + \frac{\partial E_{32}}{\partial y}\Big) + E_{32}\frac{\partial E_{32}}{\partial y} + E_{31}\frac{\partial E_{32}}{\partial x}\Big] = 0,$$

$$2\Big[E_{31}\frac{\partial q}{\partial x} + E_{32}\frac{\partial q}{\partial y} + q\Big(\frac{\partial E_{31}}{\partial x} + \frac{\partial E_{32}}{\partial y}\Big)\Big] = 0, \qquad \frac{\partial E_{32}}{\partial x} - \frac{\partial E_{31}}{\partial y} = 0.$$

Combining the first and second equalities with the fourth equality and simplifying the relations, we obtain

$$\frac{\partial q}{\partial x} + E_{31}\Big[E_{31}\frac{\partial q}{\partial x} + E_{32}\frac{\partial q}{\partial y} + q\Big(\frac{\partial E_{31}}{\partial x} + \frac{\partial E_{32}}{\partial y}\Big)\Big] + \frac{q}{2}\frac{\partial}{\partial x}(E_{31}^2 + E_{32}^2) = 0; \tag{11}$$

$$\frac{\partial q}{\partial y} + E_{32}\Big[E_{31}\frac{\partial q}{\partial x} + E_{32}\frac{\partial q}{\partial y} + q\Big(\frac{\partial E_{31}}{\partial x} + \frac{\partial E_{32}}{\partial y}\Big)\Big] + \frac{q}{2}\frac{\partial}{\partial y}(E_{31}^2 + E_{32}^2) = 0; \tag{12}$$

$$E_{31}\frac{\partial q}{\partial x} + E_{32}\frac{\partial q}{\partial y} + q\Big(\frac{\partial E_{31}}{\partial x} + \frac{\partial E_{32}}{\partial y}\Big) = 0; \tag{13}$$

$$\frac{\partial E_{32}}{\partial x} - \frac{\partial E_{31}}{\partial y} = 0. \tag{14}$$

By virtue of (13) and relation $E = -(E_{31}^2 + E_{32}^2)$, Eqs. (11) and (12) are simplified and become equations for determining the quantity $\ln q - E/2$:

$$\frac{\partial}{\partial x}\Big(\ln q - \frac{1}{2}E\Big) = 0, \qquad \frac{\partial}{\partial y}\Big(\ln q - \frac{1}{2}E\Big) = 0.$$

Integrating these equations, we obtain this quantity (and, hence, the pressure) up to a constant:

$$\ln q - E/2 = \ln h, \qquad q = h\exp(E/2), \qquad h = \text{const}. \tag{15}$$

Relations (9) and (15) yield the equation for the elastic potential, from which the potential is determined by quadrature:

$$(1+E)U'(E) = -h\exp(E/2), \quad U = -h\int(1+E)^{-1}\exp(E/2)\,dE + g, \quad g = \text{const}. \tag{16}$$

For small strains ($|E| \ll 1$), the linear approximation for the derivative of the potential and the quadratic approximation for the potential are given by

$$U' = -h(1 - E/2), \qquad U = h(E^2/4 - E) + g. \tag{17}$$

Thus, for the model considered, the antiplane strain occurs only for the elastic potential (16) [or (17) in the case of small strains].

The constant h in the expression for the elastic potential can be determined in terms of the longitudinal component P_3 of the end-load resultant. Indeed, according to (10) and (16), for the cylinder end S with the normal vector $(0, 0, 1)$ we obtain

$$P_3 = \int\limits_S p_3\, dS = \int\limits_S P_{33}\, dS = -hJ, \qquad J = \int\limits_S \exp(E/2)(1 - E)\, dS, \qquad h = -\frac{P_3}{J}.$$

For $|E| \ll 1$, the linear approximations yields

$$\exp(E/2)(1 - E) = 1 - \frac{E}{2}, \quad J = \frac{S(2 - E_*)}{2}, \quad E_* = \frac{1}{S}\int\limits_S E\, dS, \quad h = -\frac{2P_3}{S(2 - E_*)},$$

where E_* is the average value of the invariant in a cross section of the body.

In system (13), (14), we write Eq. (13) as

$$\frac{\partial E_{31}}{\partial x} + \frac{\partial E_{32}}{\partial y} + E_{31}\frac{\partial \ln q}{\partial x} + E_{32}\frac{\partial \ln q}{\partial y} = 0$$

and taking into account (15) (using the expression $E = -E_{31}^2 - E_{32}^2$ for the invariant), we eliminate the pressure from it. The resulting relation along with Eq. (14) form a system of equations for determining the independent strains, free from the elastic potential:

$$(1 - E_{31}^2)\frac{\partial E_{31}}{\partial x} + (1 - E_{32}^2)\frac{\partial E_{32}}{\partial y} - E_{31}E_{32}\Big(\frac{\partial E_{32}}{\partial x} + \frac{\partial E_{31}}{\partial y}\Big) = 0, \quad \frac{\partial E_{32}}{\partial x} - \frac{\partial E_{31}}{\partial y} = 0. \tag{18}$$

For these equations, we obtain the characteristic determinant D [6], which is a quadratic form of the quantities v_1 and v_2:

$$D = (1 - E_{31}^2)v_1^2 - 2E_{31}E_{32}v_1v_2 + (1 - E_{32}^2)v_2^2.$$

For the Sylvester conditions [7]

$$1 - E_{31}^2 > 0, \qquad (1 - E_{31}^2)(1 - E_{32}^2) - E_{31}^2E_{32}^2 > 0,$$

which can be reduced to the condition

$$E_{31}^2 + E_{32}^2 < 1, \tag{19}$$

the quadratic form is positive definite (the determinant D is positive). Consequently, the characteristic equation $D = 0$ has no real roots. In this case, system (18) is of elliptic type and the boundary-value problem with specified strains is well-posed for this system.

If the lateral surface of the cylinder is subjected to forces p_k constant along the cylinder, the relations $p_k = P_{kl}n_l$, where $(n_l) = (n_1, n_2, 0)$ is the outward normal vector, are the nonlinear system of equations for the independent strains, which hold at the contour L of the section S:

$$p_1 = P_{1l}n_l = -qn_1 + qE_{31}(E_{31}n_1 + E_{32}n_2), \quad p_2 = P_{2l}n_l = -qn_2 + qE_{32}(E_{31}n_1 + E_{32}n_2),$$

$$p_3 = P_{3l}n_l = 2q(E_{31}n_1 + E_{32}n_2) \qquad \text{on} \quad L.$$

To simplify these equations, we write the forces in the natural axes of the contour: normal (n_k), tangent (t_k), and binormal (b_k). Using the representations of the unit vectors of the natural axes and introducing the quantities E_n and E_t related to the independent strains and linear strain invariant by the formulas

$$(n_k) = (n_1, n_2, 0), \qquad (t_k) = (t_1, t_2, 0) = (-n_2, n_1, 0), \qquad (b_k) = (0, 0, 1),$$

$$E_n = E_{3k}n_k = E_{31}n_1 + E_{32}n_2, \qquad E_t = E_{3k}t_k = -E_{31}n_2 + E_{32}n_1,$$

$$E = -E_{31}^2 - E_{32}^2 = -E_n^2 - E_t^2, \tag{20}$$

$$E_{31} = E_n n_1 - E_t n_2, \qquad E_{32} = E_n n_2 + E_t n_1 \qquad \text{on} \quad L,$$

we write the natural components of the forces p_n, p_t, and p_b as

$$p_n = p_k n_k = -q(1 + E_n^2); \tag{21}$$

$$p_t = p_k t_k = -q E_n E_t, \qquad p_b = p_k b_k = 2qE_n \qquad \text{on} \quad L, \tag{22}$$

where, according to (15) and (20), the pressure at the boundary is given by

$$q = h \exp\left(-(E_n^2 + E_t^2)/2\right) \qquad \text{on} \quad L. \tag{23}$$

In this case, the quantities E_t and E_n are determined by relations (22):

$$E_t = -2p_t/p_b, \qquad E_n \exp\left(-E_n^2/2\right) = p_b \exp\left(2p_t^2/p_b^2\right)/(2h) \qquad \text{on} \quad L, \tag{24}$$

and equality (21) [after substitution of q and E_n from (23) and (24) into it] imposes a restriction on the load. For small strains, the transcendental equation in (24) is simplified and the quantities E_t and E_n are written in a linear approximation as

$$E_t = -2p_t/p_b, \qquad E_n = p_b/(2h) \qquad \text{on} \quad L. \tag{25}$$

Thus, the boundary values of the independent strains are expressed by formulas (20), in which the quantities E_t and E_n are related to the forces by (24) [by formulas (25) for small strains]. Equations (18) and conditions (20) are the boundary-value problem for the independent strains.

Based on the problem for the strains, one can obtain the problem for the displacement. By virtue of the equalities in (3), which express the independent strains in terms of the displacement:

$$2E_{31} = w_x, \qquad 2E_{32} = w_y, \tag{26}$$

the second equation in system (18) is satisfied identically and the first equation is reduced to the second-order nonlinear equation for the displacement

$$(4 - w_x^2)w_{xx} - 2w_x w_y w_{xy} + (4 - w_y^2)w_{yy} = 0. \tag{27}$$

For this equation, the ellipticity condition (19) is given by $w_x^2 + w_y^2 < 4$ and the boundary displacement is determined by formula (5) as

$$w = \int\limits_{u_0}^{u} \Big(E_{31}(u)x'(u) + E_{32}(u)y'(u)\Big)\, du + w_0 \qquad \text{on} \quad L$$

(u is a parameter).

The nonlinear equation for the displacement (27) can be reduced to a linear equation by transformation of variables. Setting $s = 2E_{31}$ and $t = 2E_{32}$, we write (26) as the Legendre transformation

$$s = w_x, \qquad t = w_y, \qquad W = xs + yt - w. \tag{28}$$

Here w and W are the generating functions of the direct and inverse transformations, respectively. This transformation allows one to pass from the physical-plane coordinates (x and y) to the coordinates of the plane of the doubled independent strains (s, t). In the process, the first derivatives of the function w in Eq. (27) are transformed into the variables s and t according to (28) and the second variables can be expressed in terms of the second derivatives of the function W. To this end, we differentiate the third equality in (28) with respect to s and t to obtain the inverse-transformation formulas

$$\begin{aligned} W_s &= x + sx_s + ty_s - w_s = x + w_x x_s + w_y y_s - w_s = x, \\ W_t &= y + sx_t + ty_t - w_t = y + w_x x_t + w_y y_t - w_t = y. \end{aligned} \tag{29}$$

Differentiating these relations with respect to x and y, we obtain two systems of linear equations:

$$W_{ss}s_x + W_{st}t_x = 1, \qquad W_{st}s_x + W_{tt}t_x = 0$$

for s_x and t_x;

$$W_{ss}s_y + W_{st}t_y = 0, \qquad W_{st}s_y + W_{tt}t_y = 1$$

for s_y and t_y. Provided the Jacobian G does not vanish,

$$G = W_{ss}W_{tt} - W_{st}^2 \neq 0,$$

the above equations yield the desired formulas for the second-order derivatives of the displacement

$$w_{xx} = s_x = W_{tt}/G, \qquad w_{xy} = s_y = t_x = -W_{st}/G, \qquad w_{yy} = t_y = W_{ss}/G. \tag{30}$$

Substituting (28) and (30) into Eq. (27), we obtain the linear differential equation for the function $W(s,t)$ with the ellipticity condition $s^2 + t^2 < 4$:

$$(4 - t^2)W_{ss} + 2stW_{st} + (4 - s^2)W_{tt} = 0. \tag{31}$$

In polar coordinates R and V of the strain plane

$$R = \sqrt{s^2 + t^2}, \qquad \tan V = t/s \qquad (s = R\cos V, \quad t = R\sin V),$$

Eq. (31) is simplified. Simple calculations show that the terms on the left side (31) are given by

$$4(W_{ss} + W_{tt}) = 4\Big(W_{RR} + \frac{1}{R}\,W_R + \frac{1}{R^2}\,W_{VV}\Big),$$

$$-(t^2W_{ss} - 2stW_{st} + s^2W_{tt}) = -RW_R - W_{VV},$$

and, hence, Eq. (31) becomes

$$4R^2W_{RR} + R(4 - R^2)W_R + (4 - R^2)W_{VV} = 0, \tag{32}$$

for which ellipticity condition is given by $R < 2$.

In the physical plane, the contour L is assumed to be determined by the equations $x = x(u)$ and $y = y(u)$. On the contour, the strains $E_{31} = E_{31}(u)$ and $E_{32} = E_{32}(u)$ and, hence, the quantities $s = s(u)$ and $t = t(u)$ (determining the contour in the strain plane) and the displacement [see (5)] are specified

$$w(u) = \int\limits_{u_0}^{u} \Big(s(u)x'(u) + t(u)y'(u)\Big)\,du + w_0.$$

Then, by virtue of (28), the function $W(u)$ is known on L which determines the boundary condition for Eq. (32):

$$W = W(u) = x(u)s(u) + y(u)t(u) - w(u) \qquad \text{on} \quad L. \tag{33}$$

Equation (32) and condition (33) constitute the boundary-value problem for W in the variables R and V.

Let the solution $W(R,V)$ of the problem be found. Then, the inverse-transformation formulas (28) and (29) written in the variables R and V

$$x = W_R\cos V - \frac{1}{R}\,W_V\sin V, \quad y = W_R\sin V + \frac{1}{R}\,W_V\cos V, \quad w = RW_R - W, \tag{34}$$

determine the displacement in the physical plane in a parametric form.

One can obtain, in an explicit form, the displacement as a function of the form $w(x,y)$. By virtue of relations (34), the Jacobian relating the variables x and y to the variables R and V can be written as

$$\frac{\partial\,(x,y)}{\partial\,(R,V)} = x_Ry_V - x_Vy_R = \frac{1}{R}\,W_{RR}(RW_R + W_{VV}) - \frac{1}{R}\,\Big(W_{RV} - \frac{1}{R}\,W_V\Big)^2.$$

The differential equation (32) implies

$$RW_R + W_{VV} = -4R^2(4 - R^2)^{-1}W_{RR}.$$

Therefore, the Jacobian becomes

$$\frac{\partial\,(x,y)}{\partial\,(R,V)} = -\frac{1}{R}\Big[\frac{4R^2}{4 - R^2}\,W_{RR}^2 + \Big(W_{RV} - \frac{1}{R}\,W_V\Big)^2\Big] \qquad (R \neq 0),$$

which with allowance for the ellipticity condition $4 - R^2 > 0$ implies that $\partial\,(x,y)/\partial\,(R,V) < 0$. Nonzero Jacobian ensures invertibility of the transformation determined by the first and second equalities in (34), i.e., the existence of the functions $R = R(x,y)$ and $V = V(x,y)$. Using these functions, one represents the displacement explicitly

$w(R,V) = w(x,y)$. Thus, to determine the displacement in the physical plane, it suffices to solve problems (32) and (33).

We consider the nonlinear equation (27) for the displacement in the physical plane. This equation admits, in particular, self-similar solutions of the form [8]

$$w = xZ(f), \qquad f = y/x, \tag{35}$$

where the function $Z(f)$ is determined from the equation

$$Z''\Big[(1+f^2)Z' - fZ - 2\sqrt{1+f^2}\Big]\Big[(1+f^2)Z' - fZ + 2\sqrt{1+f^2}\Big] = 0.$$

This equation yields the equations

$$Z'' = 0, \quad (1+f^2)Z' - fZ - 2\sqrt{1+f^2} = 0, \quad (1+f^2)Z' - fZ + 2\sqrt{1+f^2} = 0,$$

whose solutions are, respectively, given by

$$Z = A + Bf, \qquad Z = 2\sqrt{1+f^2}\,(A + \arctan f), \qquad Z = 2\sqrt{1+f^2}\,(A - \arctan f),$$

where $A = \text{const}$ and $B = \text{const}$.

According to (35), these solutions yield the displacements

$$w = Ax + By = r(A\cos v + B\sin v), \tag{36}$$

$$w = 2\sqrt{x^2+y^2}\,(A + \arctan\,(y/x)) = 2r(A+v), \quad w = 2\sqrt{x^2+y^2}\,(A - \arctan\,(y/x)) = 2r(A-v)$$

(r and v are polar coordinates of the physical plane), which can be used in solving problems.

In particular, for a circular cylinder whose section S is bounded by a circle L of radius R, the last solution in (36) determines the displacement field with the boundary displacement w_L:

$$w = 2r(w_*/(2R) - v), \qquad w_L = w_* - 2Rv, \qquad A = w_*/(2R). \tag{37}$$

Here the constant A is determined by the displacement w_* at the point of a circle with the coordinates $r = R$ and $v = 0$.

In the solution (37) dependent on polar coordinates r and v, the displacement increases with the polar radius and decreases as the polar angle increases. In this case, we have $w(0) = 0$ and $w(R,v) = w_L$. Upon passing a loop around the coordinate origin, the polar angle varies from 0 to 2π; therefore, the displacement is a multi-valued function of the coordinates. The multi-valuedness can be interpreted as follows: the cylinder is first cut by a half-plane passing through its axis, then one part is shifted with respect to the other along the cylinder axis and glued again. These displacements are typical of a screw dislocation whose axis coincides with the z axis. Thus, the solution (37) describes the screw dislocation in the cylinder. In contrast to (37), the displacement in the screw dislocation in a circular cylinder, studied in the linear theory of elasticity [4], does not depend on the polar radius.

REFERENCES

1. V. V. Novozhilov, *Theory of Elasticity* [in Russian], Sudpromgiz, Leningrad (1958).
2. J. N. Sneddon and D. S. Berry, *The Classical Theory of Elasticity*, Springer-Verlag, Berlin (1958).
3. A. I. Lur'e, *Nonlinear Theory of Elasticity* [in Russian], Nauka, Moscow (1980).
4. Yu. N. Rabotnov, *Mechanics of Deformable Solids* [in Russian], Nauka, Moscow (1988).
5. V. D. Bondar', "Modeling of Nonlinear Antiplane Strain of a Cylindrical Body," *J. Appl. Mech. Tech. Phys.*, **46**, No. 4, 539–548 (2005).
6. N. G. Petrovskii, *Lectures on Partial Differential Equations* [in Russian], Fizmatgiz, Moscow (1961).
7. G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill Company, New York (1968).
8. A. D. Polyanin, *Nonlinear Equations of Mathematical Physics: Handbook* [in Russian], Fizmatgiz, Moscow (2002).